SYNCHRONOUS MOTIONS IN A SYSTEM OF OBJECTS WITH SUPPORTING CONSTRAINTS

PMM Vol. 31, No. 4, 1967, pp. 631-642

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(*Received February 18, 1967*)

Synchronous motions of almost conservative objects with one degree of freedom interacting by way of constraints of the first and second kind [2] are considered within the framework of the general problem of dynamic systems synchronization [1]. It is assumed that the elements of the constraints of the second kind do not have degrees of freedom of their own and that the oscillations of the constraints of the first kind (i.e. of the supporting system) are accompanied by marked energy dissipation.

Periodic solutions of the rotation type are found for a system with a multidimensional rapidly rotating phase of sufficiently general form. The necessary and sufficient conditions for their stability are determined. The representability of these conditions in terms of the average energy characteristics of the motion under consideration is discussed in relation to the synchronization problem. It is shown that one of the formulations of the integral stability criterion is valid in the presence of gyroscopic forces in the supporting system.

The first group of stability conditions for synchronous states in an almost conservative system of general form and the resulting formulations of the integral criterion are obtained in [2]. A second group of stability conditions for a nonselfcontained system without supporting constraints is derived in [3]. However, in view of the applicability of the integral stability criterion, these conditions are trivial for most of the practically interesting problems subsumed by the case considered in [3].

1. **The equations of motion.** The motion of a system of n dynamic objects interacting by way of weak constraints of the first and second kind [2] will be described by a set of n pairs of "characteristic" [3] canonical variable objects q_1 and p_i ($i = 1,\ldots, n$) and by the vector column $\mathbf{z}$ consisting of the m generalized coordinates $z_1,\ldots, z_m$ required to describe the configuration of the supporting system.

The equations of motion in synchronization problems with constraints of the first kind can be conveniently constructed in the form of Rauss equations. The general Rauss kinetic potential of the system can be written as

$$L_R = R - \Pi = -\sum_{i=1}^{n} H_i + \mu L_0 + \mu^2 \ldots, \qquad L_0 = \Delta L + L^{(1)} + L^{(2)} \quad (1.1)$$

Here $\mu > 0$ is the basic small parameter of the problem characterizing the weakness of interactions between the objects, while the quantity

$$H_i = {}^1/_2\, a_i(q_i)\, p_i^2 + \Pi_i(q_i) \quad (1.2)$$

represents the "characteristic energy" (partial Hamiltonian [2]) of the i-th object). The remaining quantities appearing in (1.1) are the following characteristics of the system computed to within terms of order μ^2:

$$\mu\Delta L = \mu\Big[\sum_{i=1}^{n} \mathbf{b}_i'(q_1, \ldots, q_n)\mathbf{u}^{\cdot}p_i - \mathbf{c}'(q_1, \ldots, q_n)\mathbf{u}\Big]\Big(\mathbf{u} = \frac{\mathbf{z}}{\mu}\Big) \quad (1.3)$$

is the additional kinetic potential of the objects occasioned by small oscillations of the supporting system (the dot denotes differentiation with respect to time; the prime denotes a transposed vector, a vector row in the present instance),

$$\mu L^{(1)} = \mu\,({}^1/_2(\mathbf{u}^{\cdot})' M\mathbf{u}^{\cdot} - {}^1/_2\,\mathbf{u}'C\mathbf{u}) \quad (1.4)$$

is the kinetic potential of the supporting system, and finally

$$\mu L^{(2)} = \mu\Big[\sum_{i,\, j=1}^{n} a_{ij}(q_1, \ldots, q_n)\, p_i p_j - \Pi^{(2)}(q_1, \ldots, q_n)\Big] \quad (1.5)$$

is the kinetic potential of the elements of the constraints of the second kind [2] (it is assumed that these constraints do not have degrees of freedom of their own).

In accordance with general notions concerning the weakness of interactions in synchronization problems [1 and 2], we consider the displacements of the supporting system to be small quantities of order μ. The remaining symbols are those adopted in [2].

We make the following assumptions about the nonpotential forces corresponding to the generalized coordinates adopted above. We assume that the forces associated with the characteristic coordinates of the objects are small and partial, i.e. that

$$Q_i^* = \mu Q_i(q_i, p_i) + \mu^2 \ldots \qquad (i = 1, \ldots, n) \quad (1.6)$$

The nonpotential forces along the coordinates of the supporting system, which (by virtue of our initial assumptions about the properties of the supporting system [2]) are proportional to the displacement velocities

$$Q(\mathbf{z}, \mathbf{z}^{\cdot}) = -B\mathbf{u}^{\cdot} + \mu \ldots \qquad (Q(\mathbf{z}, 0) \equiv 0) \quad (1.7)$$

to within terms of order μ, are assumed to be small.

In (1.7) the symbol B (just as M and C in (1.4)) denotes some square $m \times m$ matrix with constant components.

We also assume that the inequality

$$(\mathbf{z}^{\cdot})'\, Q(\mathbf{z}, \mathbf{z}^{\cdot}) \leqslant 0 \quad (1.8)$$

is always fulfilled, so that the symmetric part B_c of the matrix B is associated with the positive quadratic form

$${}^1/_2(\mathbf{u}^{\cdot})' B_c \mathbf{u}^{\cdot} > 0 \quad (1.9)$$

which can be interpreted as a dissipation function.

Constructing the equations of motion of the system in the Ruass form, we obtain

$$q_i^{\cdot} - \frac{\partial H_i}{\partial p_i} = -\mu\frac{\partial}{\partial p_i}(\Delta L + L^{(2)}) + \mu^2 \ldots \qquad (i = 1, \ldots, n)$$

$$p_i^{\cdot} + \frac{\partial H_i}{\partial q_i} = \mu\Big[Q_i(q_i, p_i) + \frac{\partial}{\partial p_i}(\Delta L + L^{(2)})\Big] + \mu^2 \ldots$$

$$M\mathbf{u}^{\cdot\cdot} + B\mathbf{u}^{\cdot} + C\mathbf{u} = -\Big(\frac{d}{dt}\frac{\partial}{\partial \mathbf{u}^{\cdot}} - \frac{\partial}{\partial \mathbf{u}}\Big)\Delta L + \mu \ldots \quad (1.10)$$

For $\mu = 0$ system (1.10) yields n independent conservative second-order systems describing the motion of isolated objects without nonpotential forces. Let each of these subsystems in some domain G_i of the partial phase plane (cylinder) admit of a $2\pi/\omega_i$-periodic librational (rotational) solution of the form

$$q_i = x_i(\varphi_i, \omega_i), \quad p_i = y_i(\varphi_i, \omega_i) \quad (1.11)$$

where $\varphi_i = \omega_i t + \theta_i$ is the characteristic rapidly rotating phase and ω_i is the circular frequency dependent on the initial conditions. Then, converting to the new "phase-frequency" variables φ_i, $\omega_i (i = 1, \ldots, n)$ in Eqs. (1.10), we finally arrive at the following specific system:

$$\varphi_i^{\cdot} = \omega_i - \mu\frac{1}{k_i(\omega_i)}\Big(\frac{\partial x_i}{\partial \omega_i} Q_i + \frac{\partial L_0}{\partial \omega_i}\Big) + \mu^2 \ldots$$

$$\omega_i^{\cdot} = \mu \frac{1}{k_i(\omega_i)} \left(\frac{\partial x_i}{\partial \varphi_i} Q_i + \frac{\partial L_0}{\partial \varphi_i} \right) + \mu^2 \ldots \quad (i = 1, \ldots, n)$$

$$M\mathbf{u}^{\cdot\cdot} + B\mathbf{u}^{\cdot} + C\mathbf{u} = -\left(\frac{d}{dt} \frac{\partial}{\partial \mathbf{u}^{\cdot}} - \frac{\partial}{\partial \mathbf{u}} \right) \Delta L + \mu \ldots$$

$$k_i(\omega_i) = \frac{1}{\omega_i} \frac{dh_i(\omega_i)}{d\omega_i} = O(1), \;\; h_i(\omega_i) = H_i(x_i, y_i) \tag{1.12}$$

Here $k_i(\omega_i)$ is the slope of the skeletal curve of the i-th isolated object.

The purpose of the discussion below is to obtain the conditions of existence and stability of the synchronous solutions of system (1.12) or (which is the same thing) of system (1.10) closely related to certain synchronous solutions from family (1.11).

2. **Synchronous solutions in a system with multidimensional rapid rotations. The nonselfcontained case.** Simplifying the problem somewhat, let us consider the interaction of substantially nonlinear almost conservative objects described by the following system with a multidimensional rapidly rotating phase:

$$\varphi_i^{\cdot} = \omega_i + \mu X_i(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}, \nu t) + \mu^2 \ldots$$

$$\omega_i^{\cdot} = \mu Y_i(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}, \nu t) + \mu^2 \ldots \quad (i = 1, \ldots, n)$$

$$\mathbf{v}^{\cdot} = A\mathbf{v} + \mathbf{F}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \nu t) + \mu \ldots \tag{2.1}$$

Here $\mathbf{v}$ and $\mathbf{F}$ are N-dimensional vectors, A is a quadratic $N \times N$ matrix with constant components, the functions X_i, Y_i $(i = 1, \ldots, n)$, $\mathbf{F}$ etc. are assumed to be analytic in a certain domain G of the space of its arguments and 2π-periodic in each of the variables $\varphi_1, \ldots, \varphi_n$ and in dimensionless time $\tau = \nu t$; ν is the frequency of the external perturbation.

In some interval $0 < \mu < \mu_0$ system (2.1) admits of a synchronous solution of the form

$$\varphi_i = \tau + \alpha_i + \mu \ldots, \qquad \omega_i = \nu_i + \mu \ldots$$

$$\mathbf{v} = \mathbf{v}^{(0)} \;\; (\tau, \alpha_1, \ldots, \alpha_n, \nu_1, \nu_1, \ldots, \nu_n) + \mu \ldots \tag{2.2}$$

which is analytic in the parameter μ. This is so provided the quantity $\alpha_1, \ldots, \alpha_n$ forms a simple solution of the system

$$\nu_1 = \ldots = \nu_n = \nu$$

$$P_i(\alpha_1, \ldots, \alpha_n, \nu, \nu_1, \ldots, \nu_n) \equiv \frac{1}{2\pi} \int_0^{2\pi} (Y_i)\, d\tau = 0 \quad (i = 1, \ldots, n) \tag{2.3}$$

$$(Y_i) = Y_i(\tau + \alpha_1, \ldots, \tau + \alpha_n, \nu_1, \ldots, \nu_n, \mathbf{v}^{(0)}, \tau)$$

Here $\mathbf{v}^{(0)}$ is a solution of Eq.

$$\mathbf{v}^{(0)\cdot} = A\mathbf{v}^{(0)} + \mathbf{F}(\tau + \alpha_1, \ldots, \tau + \alpha_n, \nu_1, \ldots, \nu_n, \tau) \tag{2.4}$$

This solution is 2π-periodic in τ.

We assume here that Eq.

$$\mathbf{v}^{\cdot} = A\mathbf{v} + \Phi(t) \tag{2.5}$$

with any sufficiently smooth $2\pi/\nu$-periodic function $\Phi(t)$ in its right-hand side admits of a unique and stable $2\pi/\nu$-periodic solution $\mathbf{v}_\Phi(t)$ such that if $\max |\Phi(t)| = O(1)$, then $\max |\mathbf{v}_\Phi(t)| = O(1)$.

Turning now to the matter of the stability of the synchronous solutions determined in accordance with (2.2), (2.3) and (2.4), let us write out the equations in the variations of initial system (2.1) corresponding to these solutions,

$$\delta\varphi_i^{\cdot} = \delta\omega_i + \mu \left\{ \sum_{j=1}^{n} \left[\left(\frac{\partial X_i}{\partial \varphi_j} \right) \delta\varphi_j + \left(\frac{\partial X_i}{\partial \omega_j} \right) \delta\omega_j \right] + \left(\frac{\partial X_i}{\partial \mathbf{v}} \right) \delta\mathbf{v} \right\} + \mu^2 \ldots$$

$$\delta\omega_i^{\cdot} = \mu \left\{ \sum_{j=1}^{n} \left[\left(\frac{\partial Y_i}{\partial \varphi_j} \right) \delta\varphi_j + \left(\frac{\partial Y_i}{\partial \omega_j} \right) \delta\omega_j \right] + \left(\frac{\partial Y_i}{\partial \mathbf{v}} \right) \delta\mathbf{v} \right\} + \mu^2 \ldots$$

$$\delta \mathbf{v}^{\cdot} = A\delta \mathbf{v} + \sum_{j=1}^{n} \left(\frac{\partial \mathrm{F}}{\partial \varphi_j}\right) \delta\varphi_j + \left(\frac{\partial \mathrm{F}}{\partial \omega_j}\right) \delta\omega_j \Big] + \mu \ldots \qquad (i = 1, \ldots, n) \tag{2.6}$$

Here and below parentheses around the derivatives indicate that these are computed for the generating solution obtained from (2.2) for $\mu = 0$. Eqs. (2.6) constitute a linear system with $2\pi/\nu$-periodic coefficients whose $2n$ characteristic ("critical") indices vanish for $\mu = 0$. The stability of synchronous solutions (2.2) is ultimately determined by the signs of the real parts of the critical indices.

Limiting ourselves for this reason to the determination of the critical characteristic indices, we exclude from system (2.6) the variations of the supporting system coordinates. To do this we attempt to find them in the form

$$\delta \mathbf{v} = \sum_{i=1}^{n} [\xi_i (t, \mu) \delta\varphi_i + \eta_i (t, \mu) \delta\omega_i] \tag{2.7}$$

where ξ_i and η_i $(i = 1,\ldots, n)$ are $2\pi/\nu$-periodic vector functions of time

Substituting (2.7) into the latter Eq. of system (2.6) and making use of expressions for the derivatives $\delta\varphi_i^{\cdot}$ and $\delta\omega_i^{\cdot}$, in accordance with its first $2n$ equations, we obtain a system of nonlinear equations for determining the functions ξ_i and η_i which is the matrix analogue of the Ricatti equation. A $2\pi/\nu$-periodic solution of these equations can be sought in the form of series in powers of a small parameter

$$\xi_i = \xi_i^{(0)} + \mu\xi_i^{(1)} + \mu^2 \ldots, \qquad \eta_i = \eta_i^{(0)} + \mu\eta_i^{(1)} + \mu^2 \ldots \quad (i = 1, \ldots, n) \tag{2.8}$$

The functions $\xi_i^{(0)}$ and $\eta_i^{(0)}$ are here defined as the $2\pi/\nu$-periodic solutions of Eqs.

$$\xi_i^{(0)\cdot} = A\xi_i^{(0)} + \left(\frac{\partial \mathrm{F}}{\partial \varphi_i}\right), \quad \eta_i^{(0)\cdot} = A\eta_i^{(0)} + \left(\frac{\partial \mathrm{F}}{\partial \omega_i}\right) - \xi_i^{(0)} \tag{2.9}$$

Comparing (2.9) with (2.4), we have

$$\xi_i^{(0)} = \frac{\partial \mathbf{v}^{(0)}}{\partial \alpha_i}, \qquad \eta_i^{(0)} = \frac{\partial \mathbf{v}^{(0)}}{\partial \nu_i} + \zeta_i \qquad (i = 1, \ldots, n) \tag{2.10}$$

Here the $2\pi/\nu$-periodic vector functions ζ_i satisfy the Eqs.

$$\zeta_i^{\cdot} = A\zeta_i - \frac{\partial \mathbf{v}^{(0)}}{\partial \alpha_i} \tag{2.11}$$

and are determined as follows. Converting to dimensionless time τ in (2.4) and differentiating partially with respect to the explicitly appearing frequency ν, we obtain

$$\frac{d}{dt}\left(\frac{\partial \mathbf{v}^{(0)}}{\partial \nu}\right) = A\left(\frac{\partial \mathbf{v}^{(0)}}{\partial \nu}\right) - \frac{1}{\nu}\mathbf{v}^{(0)} \tag{2.12}$$

From this and from (2.11) we have

$$\zeta_i^{\cdot} = \nu \frac{\partial^2 \mathbf{v}^{(0)}(\tau)}{\partial \nu \, \partial \alpha_i} \qquad (i = 1, \ldots, n) \tag{2.13}$$

On eliminating the variations of the supporting system coordinates in accordance with (2.7), we arrive at a system of $2n$ equations in $\delta\varphi_i$ and $\delta\omega_i$,

$$\delta\varphi_i^{\cdot} = \delta\omega_i + \mu \sum_{j=1}^{n} \left\{ \frac{\partial (X_i)}{\partial \alpha_j} \delta\varphi_j + \left[\frac{\partial (X_i)}{\partial \nu_j} + \left(\frac{\partial X_i}{\partial \mathbf{v}}\right) \zeta_j \right] \delta\omega_j \right\} + \mu^2 \ldots \tag{2.14}$$

$$\delta\omega_i^{\cdot} = \mu \sum_{j=1}^{n} \left\{ \frac{\partial (Y_i)}{\partial \alpha_j} \delta\varphi_j + \left[\frac{\partial (Y_i)}{\partial \nu_j} + \left(\frac{\partial Y_i}{\partial \mathbf{v}}\right) \zeta_j \right] \delta\omega_j \right\} + \mu^2 \ldots \quad (i = 1, \ldots, n)$$

It is clear that the characteristic indices of system (2.14) (whose total number is $2n$) are equal to the critical characteristic indices of system (2.6). Let us make the standard substitution

$$\delta\varphi_i = e^{\lambda t} \vartheta_i, \qquad \delta\omega_i = e^{\lambda t} \psi_i \quad (i = 1, \ldots, n)$$

The characteristic index (see [3] is represented in the form

$$\lambda = \lambda_1\mu^{1/2} + \lambda_2\mu + \lambda_3\mu^{3/2} + \mu^2 \ldots \tag{2.15}$$

The problem then reduces to the determination of the conditions of existence of a $2\pi/\nu$-periodic solution of the system

$$\vartheta_i^{\cdot} = \psi_i - \mu^{1/2}\lambda_1\vartheta_i + \mu\left\{-\lambda_2\vartheta_i + \sum_{j=1}^{n}\left[\frac{\partial (X_i)}{\partial \alpha_j}\vartheta_j + \left(\frac{\partial (X_i)}{\partial \nu_j} + \left(\frac{\partial X_i}{\partial \nu}\right)\zeta_j\right)\psi_j\right]\right\} - -\mu^{3/2}\lambda_3\vartheta_i + \mu^2 \ldots$$

$$\psi_i^{\cdot} = -\mu^{1/2}\lambda_1\psi_i + \mu\left\{-\lambda_2\psi_i + \sum_{j=1}^{n}\left[\frac{\partial (Y_i)}{\partial \alpha_j}\vartheta_j + \left(\frac{\partial (Y_i)}{\partial \nu_j} + \left(\frac{\partial Y_i}{\partial \nu}\right)\zeta_j\right)\psi_j\right]\right\} - -\mu^{3/2}\lambda_3\psi_i + \mu^2 \ldots \qquad (i = 1, \ldots, n) \tag{2.16}$$

The solution of system (2.16) will in turn be sought in series form,

$$\vartheta_i = \vartheta_i^{(0)} + \mu^{1/2}\vartheta_i^{(1)} + \mu\vartheta_i^{(2)} + \mu^{3/2}\vartheta_i^{(3)} + \mu^2 \ldots$$
$$\psi_i = \psi_i^{(0)} + \mu^{1/2}\psi_i^{(1)} + \mu\psi_i^{(2)} + \mu^{3/2}\psi_i^{(3)} + \mu^2 \ldots \tag{2.17}$$

Here $\vartheta_i^{(k)}$, $\psi_i^{(k)}$ $(k = 0, 1, \ldots)$ are $2\pi/\nu$-periodic functions of time.

Beginning the process of constructing the successive approximations, we readily obtain

$$\vartheta_i^{(0)} = a_i, \quad \psi_i^{(0)} = 0, \quad \vartheta_i^{(1)} = b_i, \quad \psi_i^{(1)} = \lambda_1 a_i \quad (a_i, b_i = \text{const}) \tag{2.18}$$

From the condition of periodicity of the second approximation we arrive at a system for determining the quantities a_i,

$$\sum_{j=1}^{n}\left(\frac{\partial P_j}{\partial \alpha_j} - \lambda_1^2\delta_{ij}\right)a_j = 0 \qquad (i = 1, \ldots, n) \tag{2.19}$$

The first approximations of the characteristic indices are thus the roots of Eq.

$$\left|\frac{\partial P_i}{\partial \alpha_j} - \lambda_1^2\delta_{ij}\right| = 0 \tag{2.20}$$

A periodic third approximation exists if and only if the quantities b_i satisfy the system

$$\sum_{j=1}^{n}\left(\frac{\partial P_i}{\partial \alpha_j} - \lambda_1^2\delta_{ij}\right)b_j = \lambda_1\left\{2\lambda_2 a_i - \sum_{j=1}^{n}\left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \right.\right.$$
$$\left.\left. + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial Y_i}{\partial \nu}\right)\zeta_j(\tau)\,d\tau\right]a_j\right\} \qquad (i = 1, \ldots, n) \quad \left(R_i = \frac{1}{2\pi}\int_0^{2\pi}(X_i)\,d\tau\right) \tag{2.21}$$

The second approximations of the characteristic indices can be determined from the condition of solvability of inhomogeneous system (2.21) and are of the form

$$\lambda_2 = \sum_{i,\, j=1}^{n}\left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial Y_i}{\partial \nu}\right)\zeta_j(\tau)\,d\tau\right]a_i^* a_j \tag{2.22}$$

Here the numbers a_i^* $(i = 1, \ldots, n)$ form the solution of the system conjugate to system (2.19),

$$\sum_{j=1}^{n}\left(\frac{\partial P_j}{\partial \alpha_i} - \lambda_1^2\delta_{ij}\right)a_j^* = 0 \qquad (i = 1, \ldots, n) \tag{2.23}$$

Here we assume that all the roots of Eq. (2.20) are simple and different from zero, and that the corresponding eigenvectors a and a^* are such that the normalization condition

$$\sum_{i=1}^{n} a_i a_i^* = 1 \tag{2.24}$$

can be fulfilled.

The synchronous state in the system is asymptotically stable if

$$\lambda_1^2 < 0, \qquad \lambda_2 < 0$$

for all the roots of Eq. (2.20).

3. The selfcontained case. Representability of the stability conditions in terms of the functions P_i, R_i. Let us consider the case where there is no external single-frequency perturbation and system (2.1) is selfcontained. Then, clearly, the previously unknown frequency ν of the synchronous state must be sought in series form,

$$\nu = \nu_0 + \mu\nu_1 + \mu^2 + \ldots \tag{3.1}$$

The synchronous solution of Eqs. (2.1) has, as before, the form (2.2), although the explicit parameter ν present in the generating approximation is replaced by ν_0. A precisely similar substitution must be made in the equations for determining the parameters of generating solution (2.3). We note that the function P_i in this case satisfies the relations

$$P_i(\alpha_1 + \alpha, \ldots, \alpha_n + \alpha, \ldots) \equiv P_i(\alpha_1, \ldots, \alpha_n, \ldots) \tag{3.2}$$

where α is arbitrary. This implies that system (2.3) can generally be used to find only the quantity ν_0 and the differences between the generating phase shifts.

Investigation of the stability of the synchronous state differs in this case only in the fact that relative to the first approximations of the critical characteristic indices, Eq. (2.20) has a double zero root. Hence, relations (2.22) enable us to determine (see (2.21)) just $n - 1$ pairs of second approximations of the characteristic indices corresponding to the zero roots of Eq. (2.20). As regards the zero roots, one of them corresponds to the exact zero characteristic index of the whole system in variations (2.6), while the other corresponds to the nontrivial index which decomposes into a series in whole powers of a small parameter. This index can be computed most readily to within quantities of order μ in the following way.

As we know, the sum of characteristic indices of a system of equations with periodic coefficients is equal to the trace of the coefficient matrix averaged over a period.

Computing this quantity for system (2.14), we obtain

$$\sum_{r=1}^{n} (\lambda_r^{(1)} + \lambda_r^{(2)}) = \mu \sum_{i=1}^{n} \left[\frac{\partial R_i}{\partial \alpha_i} + \frac{\partial P_i}{\partial \nu_i} + \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \nu}\right) \zeta_i \, d\tau\right] + \mu^2 \ldots$$

$$\lambda_r^{(1,2)} = \pm \lambda_{1r}\mu^{1/2} + \lambda_{2r}\mu \pm \mu^{3/2} \ldots \qquad (r = 1, \ldots, n-1)$$

$$\lambda_n^{(1)} = \lambda_{2n}\mu + \mu^2 \ldots, \; \lambda_n^{(2)} = 0 \tag{3.3}$$

Here $\pm\lambda_{1r}$ are the roots of Eq. (2.20).

This implies that the required characteristic index can be determined to within terms of order μ from the relation

$$\lambda_{2n} = \sum_{i=1}^{n} \left[\frac{\partial R_i}{\partial \alpha_i} + \frac{\partial P_i}{\partial \nu_i} + \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \nu}\right) \zeta_i(\tau) \, d\tau\right] -$$

$$- \sum_{r=1}^{n-1} \sum_{i,\,j=1}^{n} \left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \nu}\right) \zeta_j(\tau) \, d\tau\right] a_{jr} a_{ir}^* \tag{3.4}$$

where a_r and a_r^* are the eigenvectors of systems (2.19) and (2.23) normalized in accordance with (2.24) and corresponding to the roots $\pm\lambda_{1r}$. We note, further, that by virtue of the orthogonality and normalization conditions

$$\sum_{i=1}^{n} a_{ir} a_{is}^* = \delta_{rs} \qquad (r,\, s = 1, \ldots, n) \tag{3.5}$$

the matrices $A = \|a_{jr}\|$ and $A^* = \|a_{is}^*\|$ are mutually reciprocal. Hence, the quantity $2\lambda_{2r}$ (see (2.22)) ($r = 1, \ldots, n-1$) is the r-th diagonal element of the matrix

$$A \left\| \frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \nu_j}\right) \zeta_j(\tau) \, d\tau \right\| A^{-1} \tag{3.6}$$

Since the similarity transformation does not alter the trace, it follows that

$$\lambda_{2n} = \sum_{i,\, j=1}^{n} \left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \mathbf{v}}\right) \zeta_j(\tau)\, d\tau\right] a_{jn} a_{in}^* \qquad (3.7)$$

Thus, despite the fact that the expressions for the second approximations of the characteristic indices are somewhat altered, the necessary and sufficient stability conditions remain unchanged.

Here we note that in the general case considered above investigation of stability is complicated by the necessity of double integration in accordance with relations (2.13) and (2.22). There is, however, an important special case where these integrations are easy to carry out.

Let us suppose that in the non-selfcontained case the vector function $\mathbf{F}$ in the last Eq. of system (2.1) can be written as a sum,

$$\mathbf{F} = \sum_{i=0}^{n} \mathbf{F}_i(\varphi_i, \omega_1, \ldots, \omega_n) \qquad (\varphi_0 = \nu t) \qquad (3.8)$$

and that the functions Y_i ($i = 1, \ldots, n$) are linear in the coordinates of the supporting system,

$$Y_i = Y_i^{(0)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \nu t) + \mathbf{Y}_i^{(1)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \nu t)\, \mathbf{v} \qquad (3.9)$$

The oscillations of the supporting system can then be written in the form of the superposition

$$\mathbf{v}^{(0)} = \sum_{i=0}^{n} \mathbf{v}_i^{(0)}(\tau + \alpha_i, \nu, \nu_1, \ldots, \nu_n) \qquad (\alpha_0 = 0) \qquad (3.10)$$

The components $\mathbf{v}_i^{(0)}$ can be determined from Eqs.

$$\mathbf{v}_i^{(0)\cdot} = A\mathbf{v}_i^{(0)} + \mathbf{F}_i(\tau + \alpha_i, \nu_1, \ldots, \nu_n) \qquad (3.11)$$

Thus, the equations for determining the parameters of the generating solution can be written as

$$P_i \equiv \sum_{j=0}^{n} P_{ij} + P_i^{(0)} = 0 \qquad (i = 1, \ldots, n) \qquad (3.12)$$

$$P_{ij} = \frac{1}{2\pi}\int_0^{2\pi} (Y_i^{(1)})\, \mathbf{v}_j^{(0)}\, d\tau \quad (j = 0, \ldots, n), \qquad P_i^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} (Y_i^{(0)})\, d\tau \qquad (3.13)$$

From (2.13) in this case we have

$$\zeta_i = \frac{\partial \mathbf{v}_i^{(0)}}{\partial \nu} \qquad (3.14)$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial Y_i}{\partial \mathbf{v}}\right) \zeta_j\, d\tau = \frac{\partial P_{ij}}{\partial \nu} \qquad (j = 1, \ldots, n) \qquad (3.15)$$

Finally, the second group of stability conditions can be written as

$$\sum_{i,\, j=1}^{n} \left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial P_i}{\partial \nu_j} + \frac{\partial P_{ij}}{\partial \nu}\right] a_{jr} a^*_{ir} < 0 \qquad (r = 1, \ldots, n) \qquad (3.16)$$

The same conditions are valid in the selfcontained case.

4. Average energy characteristics of synchronous motions. Turning once again to our consideration of the synchronous motions of objects with weak supported and supporting constraints, we note that the system of equations of motion in "phase-frequency" variables (1.12) can always be reduced to the form (2.1) merely by the nonsingular substitution of the variables $\mathbf{u}, \mathbf{u}^\cdot$. The equations of motion of objects then remain substantially unaltered to within terms of order μ.

The equations for the generating parameters of the synchronous state can be integrated by parts and thus reduced to

$$P_i \equiv \frac{1}{k_i(\nu_i)}\left[f_i(\nu_i) + \frac{\partial}{\partial \alpha_i}(\Lambda^{(1)} + \Lambda^{(2)} + \Delta\Lambda) - \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial \mathrm{u}'}{\partial \alpha_i} B\mathrm{u}\right) d\tau\right] = 0 \qquad (4.1)$$

$$f_i(\nu_i) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial x_i}{\partial \varphi_i} Q_i\right) d\tau \qquad (i = 1, \ldots, n) \qquad (4.2)$$

is the power of the nonpotential forces corresponding to the coordinates of the i-th object averaged over a period, and the quantities

$$\Lambda^{(1)}(\alpha_1, \ldots, \alpha_n, \nu_0, \nu_1, \ldots, \nu_n) = \frac{1}{2\pi}\int_0^{2\pi}(L^{(1)})\, d\tau$$

$$\Lambda^{(2)}(\alpha_1, \ldots, \alpha_n, \nu_1, \ldots, \nu_n) = \frac{1}{2\pi}\int_0^{2\pi}(L^{(2)})\, d\tau \qquad (4.3)$$

$$\Delta\Lambda(\alpha_1, \ldots, \alpha_n, \nu_0, \nu_1, \ldots, \nu_n) = \frac{1}{2\pi}\int_0^{2\pi}(\Delta L)\, d\tau$$

are equal to the action integral of the supporting and supported constraints and to the additional action integral of the objects [2], respectively.

Now, taking the scalar product of the equation of motion of the supporting constraints in the generating approximation,

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{\mathrm{u}}} - \frac{\partial}{\partial \mathrm{u}}\right)(L^{(1)} + \Delta L) + B\dot{\mathrm{u}} = 0 \qquad (4.4)$$

and the vector row u' and then averaging the result over the period of synchronous motions, we arrive at the following scalar identity:

$$2\Lambda^{(1)} + \Delta\Lambda = \Gamma, \qquad \Gamma = \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{u}' B_a \dot{\mathrm{u}})\, d\tau \qquad (4.5)$$

Here Γ is the virial associated with the gyroscopic forces in the supporting system and B_a is the skew-symmetric part of the matrix B.

Making use of identity (4.5), we can write out equations for determining the generating parameters of the problem in two equivalent forms,

$$P_i \equiv \frac{1}{k_i(\nu_i)}\left[f_i(\nu_i) + \frac{\partial}{\partial \alpha_i}(\Lambda^{(2)} - \Lambda^{(1)}) - \frac{1}{2\pi}\int_0^{2\pi}\left(\dot{\mathrm{u}}' B\frac{\partial \mathrm{u}}{\partial \alpha_i}\right) d\tau\right] = 0 \qquad (4.6)$$

$$P_i \equiv \frac{1}{k_i(\nu_i)}\left[f_i(\nu_i) + \frac{\partial}{\partial \alpha_i}\left(\Lambda^{(2)} + \frac{1}{2}\Delta\Lambda\right) - \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial \mathrm{u}'}{\partial \alpha_i} B_c\dot{\mathrm{u}}\right) d\tau\right] = 0 \qquad (4.7)$$

In contrast to the analogous terms in (4.1) and (4.6), the last term in relations (4.7) is determined only by the energy dissipation associated with the oscillations of the supporting system. These terms are equivalent only when there are no gyroscopic forces in the supporting system, so that $\Gamma \equiv 0$. Generally, the characteristic relation

$$\Delta\Lambda + 2\Lambda^{(1)} \equiv 0 \qquad (4.8)$$

is typical of systems with purely dissipative supporting constraints (to within terms of a higher order of smallness). On the other hand, if small oscillations of the supporting system are accompanied by the action of gyroscopic forces only ($B_c = 0$), then Eqs. (4.7) become

$$P_i \equiv \frac{1}{k_i(\nu_i)}\left[f_i(\nu_i) + \frac{\partial}{\partial \alpha_i}\left(\Lambda^{(2)} + \frac{1}{2}\Delta\Lambda\right)\right] = 0 \qquad (4.9)$$

Hence, in the case where the objects are similar in the sense that

$$k_1(\nu) \equiv \ldots \equiv k_n(\nu) \equiv k(\nu), f_1(\nu) \equiv \ldots \equiv f_n(\nu) \equiv f(\nu) \qquad (4.10)$$

it is not difficult to arrive at a formulation of the integral stability criterion: a stable synchronous state in a system with purely gyroscopic nonpotential forces along the coordinates of the supporting constraints is associated with an extremum (a maximum for $k > 0$ and a minimum for $k < 0$) of the quantity $\Lambda^{(2)} + \frac{1}{2}\Delta\Lambda$ as a function of the phase shifts $\alpha_1, \ldots, \alpha_n$. We note that in the presence of gyroscopic forces other formulations of the integral stability criterion [2] are not valid because of nonfulfillment of condition (4.8).

Finally, let us consider the energy balance equation for the system

$$\nu_0 \sum_{i=1}^{n} f_i(\nu_0) = 2\Phi, \qquad \Phi = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2} u^{\cdot\prime} B_c u^{\cdot}\right) d\tau \qquad (4.11)$$

which can be readily obtained by summing Eqs. (4.7). Here Φ is the average value of the dissipation function of the supporting constraints averaged over a period.

Without bothering to write out Eq. (2.20) for determining the stability coefficients λ_1 of the first group, we note that the conditions of representability of the second group of stability conditions in our problem reduce to the relations

$$\mathbf{b}_i = \mathbf{b}_i(q_i), \qquad \mathbf{c}_i = \mathbf{c}_i(q_i) \qquad (4.12)$$

Here the additional kinetic potential can be written as a superposition,

$$\Delta L = \sum_{i=1}^{n} \Delta L_i, \qquad \Delta L_i = b_i'(q_i)\,\mathbf{u}^{\cdot} p_i - c_i'(q_i)\,\mathbf{u} \qquad (4.13)$$

The quantity ΔL_i can be readily interpreted as the additional kinetic potential of the i-th object occasioned by the oscillations of the supporting constraints. The small oscillations of the supporting system can be written as the sum

$$\mathbf{u} = \sum_{i=1}^{n} \mathbf{u}_i(\tau + \alpha_i, \nu_0, \nu_i) \qquad (4.14)$$

each of whose terms represents the contribution of the corresponding objects and satisfies Eq.

$$M\mathbf{u}_i^{\cdot\cdot} + B\mathbf{u}_i^{\cdot} + C\mathbf{u}_i = -\left(\frac{d}{dt}\frac{\partial}{\partial \mathbf{u}^{\cdot}} - \frac{\partial}{\partial \mathbf{u}}\right)\Delta L_i \qquad (4.15)$$

We also introduce the additional kinetic potential of the i-th object occasioned by the motion of the j-th object in the generating approximation,

$$\Delta L_{ij} = b_i'(x_i)\,\mathbf{u}_j y_i - c_i'(x_i)\,\mathbf{u}_j \qquad (4.16)$$

and its average value over the period of synchronous motion,

$$\Delta\Lambda_{ij}(\alpha_i - \alpha\,, \nu_0, \nu_i, \nu_j) = \frac{1}{2\pi}\int_0^{2\pi} \Delta L_{ij}\, d\tau \qquad (4.17)$$

In addition, the interaction of certain i-th and j-th objects by way of supporting constraints is characterized by quantities related to the dissipative and gyroscopic forces,

$$\Phi_{ij}(\alpha_i - \alpha_j, \nu_0, \nu_i, \nu_j) = \frac{\nu_0}{2\pi}\int_0^{2\pi} \frac{1}{2}\mathbf{u}_i^{\cdot\prime} B_c \mathbf{u}_j^{\cdot}\, d\tau = \Phi_{ji}$$

$$\Gamma_{ij}(\alpha_i - \alpha_j, \nu_0, \nu_i, \nu_j) = \frac{\nu_0}{2\pi}\int_0^{2\pi} \mathbf{u}_i' B_a \mathbf{u}_j^{\cdot}\, d\tau = \Gamma_{ji} \qquad (4.18)$$

Clearly,

$$\Delta\Lambda = \sum_{i,j=1}^{n} \Delta\Lambda_{ij}, \quad \nu_0\Phi = \sum_{i,j=1}^{n} \Phi_{ij}, \quad \nu_0\Gamma = \sum_{i,j=1}^{n} \Gamma_{ij} \qquad (4.19)$$

The equations for determining the generating parameters of the problem now assume a form similar to (3.12),

$$P_i \equiv \sum_{j=0}^{n} \frac{1}{k_i(\nu_i)} P_{ij} = 0 \qquad (i = 1, \ldots, n) \tag{4.20}$$

Here

$$P_{ij} = \frac{1}{2} \frac{\partial}{\partial \alpha_i} (\Delta\Lambda_{ij} + \Delta\Lambda_{ji}) - 2\Phi_{ij} \quad (j = 1, \ldots, n), \quad P_{i0} = f_i(\nu_i) + \frac{\partial \Lambda^{(2)}}{\partial \alpha_i} \tag{4.21}$$

The matrix $P = \|P_{ij}\|$ naturally breaks down into a symmetric part P_c and a skew-symmetric part P_a,

$$P_a = \left\| \frac{1}{2} \frac{\partial}{\partial \alpha_i} (\Delta\Lambda_{ij} + \Delta\Lambda_{ji}) \right\|, \qquad P_c = -2 \|\Phi_{ij}\| \tag{4.22}$$

The quantities R_i (see (2.21)) in this case are

$$R_i = -\frac{1}{k_i(\nu_i)} \left\{ g_i(\nu_i) + \frac{\partial \Lambda^{(2)}}{\partial \nu_i} + \frac{\partial}{\partial \nu_i} \sum_{j=1}^{n} (P_{aij} + \Gamma_{ij}) \right\} \tag{4.23}$$

$$g_i(\nu_i) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial x_i}{\partial \omega_i} Q_i \right) d\tau \tag{4.24}$$

In writing out the second group of stability conditions we introduce the quantities

$$b_{ir} = \frac{a_{ir}^*}{k_i(\nu_i)} \tag{4.25}$$

which satisfy the orthogonality conditions (see (3.5)) with the weight $k_i(\nu_i)$,

$$\sum_{i=1}^{n} k_i(\nu_i) a_{ir} b_{is} = \delta_{rs} \tag{4.26}$$

After certain transformations (3.16) yields the second group of stability conditions,

$$\sum_{i=1}^{n} \left\{ \frac{df_i}{d\nu_0} a_{ir} + \sum_{j=1}^{n} \left[\frac{dP_{ij}}{d\nu_0} + \frac{\partial^2 \Lambda^{(2)}}{\partial \alpha_i \partial \nu_j} - \frac{\partial^2 \Lambda^{(2)}}{\partial \alpha_j \partial \nu_i} \right] a_{jr} \right\} b_{ir} < 0 \quad (r = 1, \ldots, n) \tag{4.27}$$

$$\frac{df_i}{d\nu_0} = \left(\frac{df_i(\nu_i)}{d\nu_i} \right)_{\nu_i = \nu_0}, \qquad \frac{dP_{ij}}{d\nu_0} = \left(\frac{\partial P_{ij}}{\partial \nu_i} + \frac{\partial P_{ij}}{\partial \nu_j} + \frac{\partial P_{ij}}{\partial \nu_0} \right)_{\nu_i = \nu_j = \nu_0} \tag{4.28}$$

In the case where there are no dissipative forces in the supporting system, $P_c = 0$, $a_{ir} = b_{ir}$, and these conditions become

$$\sum_{i=1}^{n} \frac{df_i}{d\nu_0} a_{ir}^2 < 0 \qquad (r = 1, \ldots, n) \tag{4.29}$$

Hence it follows that if all $df_i/d\nu_0 < 0$ and $k > 0$ (see (3.10)), then the integral criterion yields the complete system of necessary and sufficient conditions of stability of the synchronous state.

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Translated by A.Y.